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HAMILTONIAN FORM OF THE KEMMER EQUATION FOR SPINLESS BOSON

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HAMILTONIAN FORM OF THE KEMMER EQUATION FOR A SPINLESS BOSON

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Ronald R. Bousek

HAMILTONIAN FORM OF THE KEMMER EQUATION

FOR A SPINLESS BOSON

by

Ronald R. Bousek

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First Lieutenant, United States Air Force

Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE

IN

PHYSICS

United States Naval Postgraduate School

Monterey, California

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This work is accepted as fulfilling

the thesis requirements for the degree of

MASTER OF SCIENCE

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PHYSICS

from the

United States Naval Postgraduate School

ACKNOWLEDGMENT

The author is indebted to Dr. William B. Zeleny for proposing a study of the Kemmer equation as a thesis project and for his generous help and inspiration.

ABSTRACT

It is shown that for a free spinless particle described by the Kemmer equation, the Hamiltonian which Kemmer originally presented yields a velocity operator whose expectation value does not agree with the correspondence principle. Hamiltonians for a free spinless particle and a spinless particle in an electromagnetic field are constructed from the Kemmer equation. These Hamiltonians differ from the ones originally presented by Kemmer. The free particle Hamiltonian yields a velocity operator whose expectation value is consistent with the correspondence principle.

SYMBOLS AND NOTATION

Greek indices run from 0 to 3, Latin indices run from 1 to 3. Repeated upper and lower indices are summed over. The metric tensor $g^{\mu\nu}$ is given by $g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$, and all other components are zero.

$c = \hbar = 1$

$x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z) = (t, \underline{x})$

$x_\mu = (x_0, x_1, x_2, x_3) = (t, -x, -y, -z) = (t, -\underline{x})$

$\partial_\mu = \dfrac{\partial}{\partial x^\mu}$

$\partial^\mu = \dfrac{\partial}{\partial x_\mu}$

$\psi(x) = \psi(t, x, y, z)$ = wave function

$\underline{P} = (P_x, P_y, P_z)$ = momentum operator in Hilbert space

$P_0 = P^0$ = energy operator in Hilbert space

$P^\mu = (P^0, P_x, P_y, P_z)$ = energy-momentum operator

$|\psi\rangle$ = state vector in Hilbert space

$\mathcal{O}^\dagger$ = Hermitian conjugate of $\mathcal{O}$

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1. Introduction.

When Kemmer originally introduced his meson wave equation, he also derived the Hamiltonians for a free particle and a particle in an electromagnetic field. [1] For the case of a spinless particle, i.e., when the wave-function has five components, we show that the expectation value of the velocity operator obtained by using Kemmer's free particle Hamiltonian is not consistent with the correspondence principle. Another difficulty with Kemmer's Hamiltonians is that although the free particle Hamiltonian is Hermitian, the Hamiltonian for a particle in an electromagnetic field is not. The usual requirement on the Hamiltonian (or any observable) in quantum mechanics is that it be Hermitian. By requiring observables to be what we call "neo-Hermitian", we construct new Hamiltonians which differ from Kemmer's. Our free particle Hamiltonian yields a velocity operator whose expectation value agrees with the correspondence principle.

2. Free Particle Kemmer Equation.

The Kemmer equation for a free particle of mass m is

$$\left(i\,\partial_\mu \beta^\mu + m\right)\psi(x) = 0 \tag{1}$$

where the β^μ satisfy the relation,

$$\beta^\mu\beta^\nu\beta^\tau + \beta^\tau\beta^\nu\beta^\mu = \beta^\mu g^{\nu\tau} + \beta^\tau g^{\nu\mu} \tag{2}$$

For a spinless particle the β^μ are 5 X 5 matrices. We shall use the following (non-unique) representation of the β^μ:

$$\beta^0 = \begin{bmatrix} 0&0&0&0&-1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ -1&0&0&0&0 \end{bmatrix} \qquad \beta^1 = \begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&-1&0&0&0 \end{bmatrix}$$

$$\beta^2 = \begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&-1&0&0 \end{bmatrix} \qquad \beta^3 = \begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&-1&0 \end{bmatrix} \tag{3}$$

Hence ψ is a five-component column symbol. When the representation (3) is inserted into (1), it is easily seen that $\psi(x)$ consists of a scalar, which satisfies the Klein-Gordon equation, and its four-gradient.

It can be shown that

$$\partial_\mu\left(\bar{\psi}\beta^\mu\psi\right) = 0$$

where

$$\bar{\psi} = \psi^+\left(2\beta^{0^2} - 1\right)$$

Therefore the conserved four-vector current j^μ is given by

$$j^\mu = \bar{\psi}\beta^\mu\psi$$

In particular, the probability density j^o is

$$j^o = \bar{\psi}\beta^o\psi = \psi^+\beta^o\psi$$

since
$$\begin{aligned}\bar{\psi}\beta^o &= \psi^+(2\beta^{o2}-1)\beta^o \\ &= \psi^+(2\beta^{o3}-\beta^o) \\ &= \psi^+(2\beta^o-\beta^o) \\ &= \psi^+\beta^o\end{aligned}$$

The spin operator for the Kemmer equation is given by

$$S_k = i\left[\beta^i, \beta^j\right] \qquad (i, j, k \text{ cyclic}) \tag{4}$$

where $S_1 = S_x$, $S_2 = S_y$, $S_3 = S_z$

It will be convenient for the discussion which follows to rewrite (1) in terms of operators and state vectors in a Hilbert space. Then in the Heisenberg picture,

$$\psi(x) = \langle x|\psi\rangle$$

and

$$i\partial_\mu\langle x|\psi\rangle = \langle x|P_\mu|\psi\rangle$$

where the ket vector $|\psi\rangle$ is the state vector in Hilbert space, and it can be written

$$|\psi\rangle = \begin{bmatrix}|\psi_0\rangle \\ |\psi_1\rangle \\ |\psi_2\rangle \\ |\psi_3\rangle \\ |\psi_4\rangle\end{bmatrix}$$

so that $|\psi\rangle$ is actually a column symbol composed of five vectors in Hilbert space.

The components of the four-momentum operator P_μ are given by

$$P_\mu = (P_0, P_1, P_2, P_3) = (P_0, -P_x, -P_y, -P_z) = (P_0, -\underline{P})$$

and

$$P^\mu = (P^0, P^1, P^2, P^3) = (P_0, P_x, P_y, P_z) = (P_0, \underline{P})$$

where P_0 is the Hermitian energy operator in Hilbert space, and P_x, P_y, and P_z are the Hermitian operators for the momentum in Hilbert space.

Thus we can write

$$(P_\mu \beta^\mu + m)|\psi\rangle = 0 \tag{5}$$

as the Kemmer equation in Hilbert space, and (1) is just the x-representation of (5).

where P_0 is the Hermitian energy operator in Hilbert space, and P_x, P_y, and P_z are the Hermitian operators for the momentum in Hilbert space.

Thus we can write

$$\left(P_\mu \beta^\mu + m\right)|\psi\rangle = 0 \qquad (5)$$

as the Kemmer equation in Hilbert space, and (3) is just the x-representation of (5).

3. Hamiltonian Form of the Free Particle Kemmer Equation.

The Hamiltonian form of (5) may be obtained as follows. Multiply (5) on the left by $P_\sigma \beta^\sigma \beta^\nu$:

$$\left(P_\sigma P_\mu \beta^\sigma \beta^\nu \beta^\mu + m P_\sigma \beta^\sigma \beta^\nu\right)|\psi\rangle = 0 \tag{6}$$

This can be rewritten as

$$\left(P_\sigma P_\mu \beta^\mu \beta^\nu \beta^\sigma + m P_\sigma \beta^\sigma \beta^\nu\right)|\psi\rangle = 0 \tag{7}$$

Now add (6) and (7) to obtain

$$\left[P_\sigma P_\mu \left(\beta^\sigma \beta^\nu \beta^\mu + \beta^\mu \beta^\nu \beta^\sigma\right) + 2m P_\mu \beta^\mu \beta^\nu\right]|\psi\rangle = 0$$

so, $$\left[P_\sigma P_\mu \left(\beta^\sigma g^{\nu\mu} + \beta^\mu g^{\nu\sigma}\right) + 2m P_\mu \beta^\mu \beta^\nu\right]|\psi\rangle = 0$$

or $$\left(P^\nu P_\sigma \beta^\sigma + P^\nu P_\mu \beta^\mu + 2m P_\mu \beta^\mu \beta^\nu\right)|\psi\rangle = 0$$

This can be written as

$$\left(P^\nu P_\mu \beta^\mu + m P_\mu \beta^\mu \beta^\nu\right)|\psi\rangle = 0$$

By (5),

$$P_\mu \beta^\mu |\psi\rangle = -m|\psi\rangle$$

so

$$\left(-P^\nu + P_\mu \beta^\mu \beta^\nu\right)|\psi\rangle = 0$$

or

$$P^\nu|\psi\rangle = P_\mu \beta^\mu \beta^\nu |\psi\rangle$$

For $\nu = 0$,

$$\begin{aligned} P^0|\psi\rangle &= P_\mu \beta^\mu \beta^0 |\psi\rangle \\ &= \left(P_0 \beta^{0^2} + P_j \beta^j \beta^0\right)|\psi\rangle \end{aligned}$$

or $$\left[P^0\left(1-\beta^{0^2}\right) - P_j \beta^j \beta^0\right]|\psi\rangle = 0 \tag{8}$$

Multiply (5) on the left by β^0 to obtain

$$\left(P_0 \beta^{0^2} + P_j \beta^0 \beta^j + m \beta^0\right) |\psi\rangle = 0 \tag{9}$$

The β^{0^2} term in (8) may now be eliminated by adding (8) and (9). The result is

$$\left(P_0 + P_j \beta^0 \beta^j - P_j \beta^j \beta^0 + m \beta^0\right) |\psi\rangle = 0 \tag{10}$$

or

$$P_0 |\psi\rangle = \left(P_j \left[\beta^j, \beta^0\right] - m \beta^0\right) |\psi\rangle \tag{11}$$

Let the quantity H' be defined by the equation

$$H' \equiv P_j \left[\beta^j, \beta^0\right] - m \beta^0 \tag{12}$$

Eq. (11) can now be written

$$P_0 |\psi\rangle = H' |\psi\rangle \tag{13}$$

At this point we are tempted to call H' the Hamiltonian for a free particle obeying the Kemmer equation. Indeed, Kemmer in his original paper took H' to be the Hamiltonian. The form (13) seems to imply this. We shall examine this more closely later.

Note that (13) is not equivalent to the original equation (5) because in the steps to obtain (13) we have multiplied by the singular matrix β^0 and by $P_\tau \beta^\tau \beta^\nu$. These steps are irreversible since β^0 has no inverse.

The part of the Kemmer equation which was "lost" in obtaining (13) can be obtained in the following manner: Multiply (10) on the left by β^0 to obtain

$$\left(P_0 \beta^0 + P_j \beta^{0^2} \beta^j + m \beta^{0^2}\right) |\psi\rangle = 0 \tag{14}$$

where we have used $\beta^0\beta^j\beta^0 = 0$ which comes from (2). Now the Kemmer equation can be written as

$$\left(P_0\beta^0 + P_j\beta^j + m\right)|\psi\rangle = 0 \qquad (15)$$

To find the "lost" part, we subtract (14) from (15) and obtain

$$\left(P_j\beta^j - P_j\beta^{0^2}\beta^j - m\beta^{0^2} + m\right)|\psi\rangle = 0 \qquad (16)$$

Thus, (16) and (13) are together equivalent to the Kemmer equation. Note that (16) does not contain P_0, so in the x-representation no time derivatives appear. Hence (16) may be taken as an initial or subsidiary condition.

Eq. (16) can be written in terms of H' as follows:

By (2)

$$\beta^{0^2}\beta^j = -\beta^j\beta^{0^2} + \beta^j$$

so (16) becomes

$$\left[P_j\beta^j - P_j\left(-\beta^j\beta^{0^2} + \beta^j\right) - m\beta^{0^2} + m\right]|\psi\rangle = 0$$

or

$$\left(P_j\beta^j\beta^{0^2} - m\beta^{0^2} + m\right)|\psi\rangle = 0 \qquad (17)$$

Since $\beta^0\beta^j\beta^0 = 0$, (17) can be written

$$\left(P_j\beta^j\beta^{0^2} - P_j\beta^0\beta^j\beta^0 - m\beta^{0^2} + m\right)|\psi\rangle = 0$$

or

$$\left[\left(P_j\left[\beta^j, \beta^0\right] - m\beta^0\right)\beta^0 + m\right]|\psi\rangle = 0 \qquad (18)$$

The quantity in parentheses is H', so (18) becomes

$$\left(H'\beta^0 + m\right)|\psi\rangle = 0 \qquad (19)$$

Thus, the subsidiary condition (19) together with (13) is equivalent to the Kemmer equation.

Because of (19) we can write

$$P_0|\psi\rangle = \left[H' + A\left(H'\beta^0 + m\right)\right]|\psi\rangle \tag{20}$$

where A can be any arbitrary matrix operator.

The problem now is to find an A such that the operator on the right side of (20) can properly be called the Hamiltonian. With this in mind, let us now consider the properties which any dynamical variable must have.

From the form of the probability density j^0, we are led to take the expectation value of an observable α to be

$$\langle\alpha\rangle = \frac{\int \psi^{+}\beta^0\alpha\,\psi\, d^3_x}{\int \psi^{+}\beta^0\,\psi\, d^3_x}$$

in the x-representation. Thus, in Hilbert space the equation,

$$\langle\alpha\rangle = \frac{\langle\psi|\beta^0\alpha|\psi\rangle}{\langle\psi|\beta^0|\psi\rangle}$$

gives the expectation value of the observable α in the state $|\psi\rangle$.

The requirement that the expectation value be real yields

$$\begin{aligned}\langle\psi|\beta^0\alpha|\psi\rangle &= \langle\psi|\beta^0\alpha|\psi\rangle^{+}\\ &= \langle\psi|(\beta^0\alpha)^{+}|\psi\rangle\\ &= \langle\psi|\alpha^{+}\beta^0|\psi\rangle\end{aligned}$$

where we have used the fact that β^0 is Hermitian.

This suggests that a requirement on an observable is that it must obey the relation,

$$\begin{aligned}\beta^0\alpha &= \left(\beta^0\alpha\right)^{+}\\ &= \alpha^{+}\beta^0\end{aligned} \tag{21}$$

Hereafter we will call an operator α obeying (21) a "neo-Hermitian" operator.

In addition to having a real expectation value, a neo-Hermitian operator has real eigenvalues. This can be shown as follows: Suppose α is neo-Hermitian and its eigenvalue equation is

$$\alpha|a\rangle = a|a\rangle$$

where a is the eigenvalue corresponding to $|a\rangle$. Then,

$$\langle a|\beta^0\alpha|a\rangle = a\langle a|\beta^0|a\rangle \tag{22}$$

Since $\beta^{0\dagger}=\beta^0$, the quantity $\langle a|\beta^0|a\rangle$ is real. Likewise, since α is neo-Hermitian, the left side of (22) is real. Therefore, the eigenvalue a must be real.

Thus, a neo-Hermitian operator satisfies the usual necessary conditions on an observable, i.e., its expectation value is real, and its eigenvalues are real.

If (21) is written out explicitly in matrix form in the representation (3), it can be seen that a neo-Hermitian matrix must have the form,

$$\begin{bmatrix} \alpha_{00} & 0 & 0 & 0 & \alpha_{04} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{40} & 0 & 0 & 0 & \alpha_{00}^{\dagger} \end{bmatrix}$$

where α_{04} and α_{40} are real (or Hermitian if they are Hilbert operators). The dots denote arbitrary matrix elements which are undetermined by (21).

Thus, we see that when working with the Kemmer equation, we are led to require that dynamical variables be represented by neo-Hermitian operators as opposed to the usual case where dynamical variables are taken to be Hermitian. Note that since P_μ is a Hermitian operator in Hilbert space, it commutes with β^0, and so P_μ is also neo-Hermitian in the 5 X 5 space.

If we write out H' as a matrix, we obtain

$$H' = \begin{bmatrix} 0 & -p_1 & -p_2 & -p_3 & m \\ -p_1 & & & & \\ -p_2 & & 0 & & \\ -p_3 & & & & \\ m & & & & \end{bmatrix}$$

We see that H' is Hermitian in the usual sense, but it is not neo-Hermitian because the elements $H'_{01}, H'_{02}, H'_{03}$ are non-zero. It is possible to make the operator on the right side of (20) neo-Hermitian by the proper choice of the matrix A, i.e., we require that

$$H \equiv H' + A\left(H'\beta^0 + m\right)$$

be neo-Hermitian if it is to be the Hamiltonian. If we use the condition,

$$\beta^0 H = \left(\beta^0 H\right)^\dagger \tag{23}$$

and write this out explicitly in the 5 X 5 matrices, it is easy to show that A must be of the form,

$$A = \begin{bmatrix} \cdot & \frac{1}{m}p_1 & \frac{1}{m}p_2 & \frac{1}{m}p_3 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & 0 & 0 & \cdot \end{bmatrix} \tag{24}$$

where the dots represent arbitrary matrix elements which are not determined by (23). Let us set these arbitrary elements equal to zero. If this is done, then A can be expressed in terms of the β^μ matrices as follows:

$$A = \frac{1}{m}\, p_j \beta^0 \beta^j \tag{25}$$

The equivalence of (24) and (25) is easily seen when it is noted that

$$\beta^0\beta^1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & & & & \\ 0 & & 0 & & \\ 0 & & & & \\ 0 & & & & \end{bmatrix} \qquad \beta^0\beta^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & & & & \\ 0 & & 0 & & \\ 0 & & & & \\ 0 & & & & \end{bmatrix}$$

$$\beta^0\beta^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & & & & \\ 0 & & 0 & & \\ 0 & & & & \\ 0 & & & & \end{bmatrix}$$

Thus, we shall take as the free particle Hamiltonian,

$$H = H' + \frac{1}{m} P_j \beta^0\beta^j (H'\beta^0 + m)$$

which can be rewritten as follows:

$$H = H' + \frac{1}{m} P_j \beta^0\beta^j \left(P_i [\beta^i, \beta^0] \beta^0 - m {\beta^0}^2 + m \right)$$

Now $\beta^0\beta^i\beta^0 = 0$, so

$$H = H' + \frac{1}{m} P_j P_i \beta^0\beta^j\beta^i{\beta^0}^2 + P_j \beta^0\beta^j$$

From (2) $\beta^i{\beta^0}^2 = -{\beta^0}^2\beta^i + \beta^i$, so

$$H = H' + \frac{1}{m} P_j P_i \beta^0\beta^j\beta^i + P_j \beta^0\beta^j \tag{26}$$

When the expression for H' is inserted,

$$H = P_j \beta^j\beta^0 + \frac{1}{m} P_i P_j \beta^0\beta^i\beta^j - m\beta^0 \tag{27}$$

To write out H in its matrix form, we can make use of the relation,

$$\beta^\mu\beta^\nu\beta^\sigma = 0 \qquad (\mu, \nu, \sigma \quad \text{all different})$$

which is satisfied for our particular representation (3). Thus, for the representation (3), $\beta^0\beta^i\beta^j = 0 \quad (i \neq j)$ can be used in (27).

In matrix form,

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{m}(\underline{P}^2+m^2) \\ -P_1 & & & & \\ -P_2 & & 0 & & \\ -P_3 & & & & \\ m & & & & \end{bmatrix} \tag{28}$$

where $\underline{P}^2 = \underline{P}\cdot\underline{P} = P_x^{\,2} + P_y^{\,2} + P_z^{\,2} = -P_j P^j$

Thus, the Hamiltonian form of the Kemmer equation is

$$P_0 |\psi\rangle = H|\psi\rangle \tag{29}$$

where H is given by (27). The Hamiltonian form (29) and the subsidiary condition (19) are together equivalent to the Kemmer equation.

When the subsidiary condition is written in matrix form, it becomes

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & P_1 \\ 0 & 0 & m & 0 & P_2 \\ 0 & 0 & 0 & m & P_3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} |\psi_0\rangle \\ |\psi_1\rangle \\ |\psi_2\rangle \\ |\psi_3\rangle \\ |\psi_4\rangle \end{bmatrix} = 0 \tag{30}$$

From (30) we see that the subsidiary condition can be expressed by the equation,

$$|\psi_i\rangle = -\frac{1}{m} P_i |\psi_4\rangle \qquad (i=1,2,3) \tag{31}$$

When (29), the Hamiltonian form, is written out as a matrix, the bottom row yields the relation,

$$|\psi_0\rangle = \frac{1}{m} P_0 |\psi_4\rangle \tag{32}$$

Eq. (29) also yields the relation,

$$-P_i |\psi_0\rangle = P_0 |\psi_i\rangle \qquad (i = 1,2,3)$$

which is consistent with (31) and (32). The top row of (29) gives

$$\frac{1}{m}\left(\underline{P}^2+m^2\right)|\psi_4\rangle = P_0|\psi_0\rangle$$

which together with (32) yields the Klein-Gordon equation,

$$\left(\underline{P}^2+m^2\right)|\psi_4\rangle = P_0^2|\psi_4\rangle$$

In the x-representation the eigenvalue equation for the Hamiltonian leads to solutions of the form

$$\psi(x) = u(p)\, e^{-i p^\mu x_\mu}$$

for states which are simultaneously eigenstates of the energy and momentum. The eigenvalues and eigenvectors of H are

$$E = \pm\mathcal{E} \quad : \qquad u_\pm(p) = \frac{1}{\sqrt{2m\mathcal{E}}}\begin{bmatrix} \pm\mathcal{E} \\ -p_1 \\ -p_2 \\ -p_3 \\ m \end{bmatrix} \qquad u_\pm^\dagger \beta^0 u_\pm = \pm 1$$

where $\mathcal{E} \equiv \sqrt{p^2+m^2}$

and

$$E = 0 \quad : \qquad v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$v_i^\dagger \beta^0 v_i = 0$$

Since the $u_\pm$ also satisfy the subsidiary condition, they are solutions of the Kemmer equation. The v_i are not solutions to the Kemmer equation.

4. The Velocity Operator.

We have shown that the quantity H' should not be taken as the Hamiltonian since it is not neo-Hermitian. As a further demonstration that it should not be taken as the Hamiltonian, we shall show that the expectation value of the velocity operator $\underline{\dot{X}} = (\dot{X}^1, \dot{X}^2, \dot{X}^3) = (\dot{X}, \dot{Y}, \dot{Z})$, when calculated by using H' as the free particle Hamiltonian, is not the correct value.

If H' were the Hamiltonian, we would have

$$\begin{aligned}\dot{X}^K &= -i\left[X^K, H'\right] \\ &= -i\left[X^K, P_j[\beta^j, \beta^0] - m\beta^0\right] \\ &= -i\left[X^K, P_j[\beta^j, \beta^0]\right]\end{aligned}$$

since X^K is a Hilbert operator and commutes with the β^μ .

$$\begin{aligned}\dot{X}^K &= -i P_j\left[X^K, [\beta^j, \beta^0]\right] - i\left[X^K, P_j\right][\beta^j, \beta^0] \\ &= -i\left[X^K, P_j\right][\beta^j, \beta^0]\end{aligned}$$

Now $[X^K, P_j] = -i g^K{}_j$, so

$$\dot{X}^K = [\beta^0, \beta^K]$$

We shall calculate the expectation value of $\dot{X}^K$ in a state which is a simultaneous eigenstate of P_0 and $\underline{P}$. Then Eqs. (31) and (32) yield

$$|\psi_i\rangle = -\frac{1}{m} P_i|\psi_4\rangle = -\frac{1}{m} p_i|\psi_4\rangle \qquad (i = 1, 2, 3)$$

and

$$|\psi_0\rangle = \frac{1}{m} P_0|\psi_4\rangle = \frac{1}{m} E|\psi_4\rangle$$

where p_i is the eigenvalue of P_i, and E is the energy of the particle in the state $|\psi\rangle$. We have previously shown that $E = \pm\sqrt{p^2 + m^2}$ for solutions of the free particle Kemmer equation.

Thus, $$\langle \dot{X}^k \rangle = \frac{\langle\psi|\beta^0[\beta^0,\beta^k]|\psi\rangle}{\langle\psi|\beta^0|\psi\rangle}$$

$$= \frac{\langle\psi|\beta^{0^2}\beta^k|\psi\rangle}{\langle\psi|\beta^0|\psi\rangle}$$

Now in our representation of the β^μ

$$\langle\psi|\beta^{0^2}\beta^k|\psi\rangle = -\langle\psi_4|\psi_k\rangle$$
$$= \frac{1}{m} p_k \langle\psi_4|\psi_4\rangle$$

and

$$\langle\psi|\beta^0|\psi\rangle = -\langle\psi_0|\psi_4\rangle - \langle\psi_4|\psi_0\rangle$$
$$= -\frac{2}{m} E\langle\psi_4|\psi_4\rangle$$

Thus,

$$\langle \dot{X}^k \rangle = -\frac{p_k}{2E} = \frac{p^k}{2E} \tag{33}$$

Now the k-component of the classical velocity for a free particle is p^k/E , so (33) is not consistent with the correspondence principle. Hence we conclude that H' can not be taken to be the Hamiltonian.

If we calculate $\langle \dot{X}^k \rangle$ by using H as the Hamiltonian, we obtain the correct results:

$$\dot{X}^k = -i[X^k, H]$$
$$= -i[X^k, P_j\beta^j\beta^0 + \frac{1}{m}P_iP_j\beta^0\beta^i\beta^j - m\beta^0]$$
$$= -i[X^k, P_j]\beta^j\beta^0 - \frac{i}{m}[X^k, P_iP_j]\beta^0\beta^i\beta^j$$
$$= -\beta^k\beta^0 - \frac{i}{m}[X^k, P_iP_j]\beta^0\beta^i\beta^j$$
$$= -\beta^k\beta^0 - \frac{i}{m}P_i[X^k, P_j]\beta^0\beta^i\beta^j - \frac{i}{m}[X^k, P_i]P_j\beta^0\beta^i\beta^j$$
$$= -\beta^k\beta^0 - \frac{1}{m}P_i\beta^0\beta^i\beta^k - \frac{1}{m}P_j\beta^0\beta^k\beta^j$$

Now in our representation,

$$\beta^0 \beta^i \beta^j = 0 \qquad (i \neq j)$$

so

$$\dot{X}^K = -\beta^K \beta^0 + \frac{2}{m} P^K \beta^0 \beta^{K^2} \tag{34}$$

and

$$\langle \dot{X}^K \rangle = \frac{\langle \psi | \beta^0 \dot{X}^K | \psi \rangle}{\langle \psi | \beta^0 | \psi \rangle}$$

$$= \frac{2}{m} P^K \frac{\langle \psi | \beta^{0^2} \beta^{K^2} | \psi \rangle}{\langle \psi | \beta^0 | \psi \rangle}$$

In our representation (3)

$$\beta^0 \beta^{K^2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ & & & & 0 \\ & & 0 & & 0 \\ & & & & 0 \\ & & & & 0 \end{bmatrix} \qquad (K = 1, 2, 3)$$

and so

$$\beta^{0^2} \beta^{K^2} = \begin{bmatrix} 0 & & & & \\ 0 & & & & \\ 0 & & 0 & & \\ 0 & & & & \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

therefore

$$\langle \psi | \beta^{0^2} \beta^{K^2} | \psi \rangle = - \langle \psi_4 | \psi_4 \rangle$$

We now have

$$\langle \dot{X}^K \rangle = \frac{-\frac{2}{m} P^K \langle \psi_4 | \psi_4 \rangle}{-\frac{2}{m} E \langle \psi_4 | \psi_4 \rangle}$$

$$= \frac{1}{E} P^K$$

which is the correct value.

It can easily be shown that the velocity operator given by (34) is neo-Hermitian:

$$\beta^0 \dot{X}^K = \frac{2}{m} P^K \beta^{0^2} \beta^{K^2}$$

In our representation $\beta^{0\dagger} = \beta^0$, and $\beta^{K\dagger} = -\beta^K$. Thus,

$$
\begin{aligned}
(\beta^0 \dot{X}^K)^\dagger &= \frac{2}{m} P^K \beta^{K^2} \beta^{0^2} \\
&= \frac{2}{m} P^K (-\beta^0 \beta^{K^2} - \beta^0) \beta^0 \\
&= \frac{2}{m} P^K (-\beta^0 \beta^{K^2} \beta^0 - \beta^{0^2}) \\
&= \frac{2}{m} P^K \left[-\beta^0 (-\beta^0 \beta^{K^2} - \beta^0) - \beta^{0^2}\right] \\
&= \frac{2}{m} P^K \beta^{0^2} \beta^{K^2} \\
&= \beta^0 \dot{X}^K
\end{aligned}
$$

and so $\dot{X}^K$ is neo-Hermitian.

5. The Total Angular Momentum.

The z-component of the total angular momentum for a particle obeying the Kemmer equation is given by

$$\begin{aligned} J_z &= L_z + S_z \\ &= X^1 P^2 - X^2 P^1 + i\left[\beta^1, \beta^2\right] \end{aligned} \tag{35}$$

where we have used (4) as the spin operator. For a free particle, the total angular momentum must be a constant of the motion. This can be verified as follows:

$$\begin{aligned} \dot{J}_z &= -i\left[J_z, H\right] \\ &= -i\Big[\left(X^1 P^2 - X^2 P^1 + i\left[\beta^1, \beta^2\right]\right), \left(P_j \beta^j \beta^0 \right. \\ &\qquad \left. - m\beta^0 + \frac{1}{m} P_i P_j \beta^0 \beta^i \beta^j\right)\Big] \end{aligned} \tag{36}$$

To evaluate this commutator, consider the following:

$$\begin{aligned} \left[X^i P^k, P_j \beta^j \beta^0\right] &= \left[X^i P^k, P_j\right] \beta^j \beta^0 \\ &= \left[X^i, P_j\right] P^k \beta^j \beta^0 \\ &= -i\, g^i{}_j\, P^k \beta^j \beta^0 \\ &= -i\, P^k \beta^i \beta^0 \end{aligned}$$

so we see that

$$\left[X^1 P^2, P_j \beta^j \beta^0\right] = -i\, P^2 \beta^1 \beta^0 \tag{37}$$

and

$$\left[X^2 P^1, P_j \beta^j \beta^0\right] = -i\, P^1 \beta^2 \beta^0 \tag{38}$$

Now

$$\begin{aligned} &\left[X^l P^k, \frac{1}{m} P_i P_j \beta^0 \beta^i \beta^j\right] \\ &= \frac{1}{m} P^k P_i \left[X^l, P_j\right] \beta^0 \beta^i \beta^j \\ &\qquad + \frac{1}{m} P^k \left[X^l, P_i\right] P_j \beta^0 \beta^i \beta^j \end{aligned}$$

$$= -\frac{i}{m} P^k P_i \beta^0 \beta^i \beta^l - \frac{i}{m} P^k P_j \beta^0 \beta^l \beta^j$$
$$= \frac{2i}{m} P^k P^l \beta^0 \beta^{l^2}$$

where we have used $\beta^0 \beta^i \beta^j = 0 \quad (i \neq j)$ which holds in our representation of the β^μ . Thus

$$\left[X^1 P^2 , \frac{1}{m} P_i P_j \beta^0 \beta^i \beta^j \right] = \frac{2i}{m} P^1 P^2 \beta^0 \beta^{1^2} \tag{39}$$

and

$$\left[X^2 P^1 , \frac{1}{m} P_i P_j \beta^0 \beta^i \beta^j \right] = \frac{2i}{m} P^1 P^2 \beta^0 \beta^{2^2} \tag{40}$$

Also, $\left[[\beta^1, \beta^2] , P_j \beta^j \beta^0 \right]$
$$= P_j \left([\beta^1, \beta^2] \beta^j \beta^0 - \beta^j \beta^0 [\beta^1, \beta^2] \right)$$
$$= P_j \left(\beta^1 \beta^2 \beta^j \beta^0 - \beta^2 \beta^1 \beta^j \beta^0 - \beta^j \beta^0 \beta^1 \beta^2 + \beta^j \beta^0 \beta^2 \beta^1 \right)$$
$$= P_2 \beta^1 \beta^{2^2} \beta^0 - P_1 \beta^2 \beta^{1^2} \beta^0$$
$$= P_2 \beta^1 \left(-\beta^0 \beta^{2^2} - \beta^0 \right) - P_1 \beta^2 \left(-\beta^0 \beta^{1^2} - \beta^0 \right)$$
$$= -P_2 \beta^1 \beta^0 + P_1 \beta^2 \beta^0 \tag{41}$$

$$\left[[\beta^1, \beta^2] , \beta^0 \right] = 0 \tag{42}$$

and

$$\left[[\beta^1, \beta^2] , P_i P_j \beta^0 \beta^i \beta^j \right] = P_i P_j \left[[\beta^1, \beta^2] , \beta^0 \beta^i \beta^j \right]$$
$$= P_i P_j \left(\beta^1 \beta^2 \beta^0 \beta^i \beta^j - \beta^2 \beta^1 \beta^0 \beta^i \beta^j \right.$$
$$\left. - \beta^0 \beta^i \beta^j \beta^1 \beta^2 + \beta^0 \beta^i \beta^j \beta^2 \beta^1 \right)$$
$$= P_i P_j \left(-\beta^0 \beta^i \beta^j \beta^1 \beta^2 + \beta^0 \beta^i \beta^j \beta^2 \beta^1 \right)$$
$$= P_1 P_1 \beta^0 \beta^1 \beta^2 - P_2 P_2 \beta^0 \beta^2 \beta^1$$
$$= 0 \tag{43}$$

To obtain (41), (42), and (43) we have used $\beta^\mu \beta^\nu \beta^\sigma = 0$ for μ, ν, σ all different, which holds in our representation of the β^μ.

When we put in expressions (37) - (43) into (36), we obtain

$$\begin{aligned}\dot{J}_z &= -P^2\beta^1\beta^0 + P^1\beta^2\beta^0 + \frac{2}{m} P^1 P^2 \beta^0 {\beta^1}^2 \\ &\quad - \frac{2}{m} P^1 P^2 \beta^0 {\beta^2}^2 - P_2 \beta^1 \beta^0 + P_1 \beta^2 \beta^0 \\ &= \frac{2}{m} P^1 P^2 \left(\beta^0 {\beta^1}^2 - \beta^0 {\beta^2}^2\right) \end{aligned} \tag{44}$$

Now in our representation of the β^μ,

$$\beta^0 {\beta^1}^2 = \beta^0 {\beta^2}^2 = \beta^0 {\beta^3}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ & & & & 0 \\ & & 0 & & 0 \\ & & & & 0 \\ & & & & 0 \end{bmatrix}$$

so (44) becomes

$$\dot{J}_z = 0$$

as required.

6. Particle in an Electromagnetic Field.

For a particle in an electromagnectic field we make the replacement

$$P_\mu \longrightarrow P_\mu - e A_\mu \equiv \mathcal{P}_\mu$$

Where A_μ is the operator for the electromagnetic four-potential.

Eq. (5) now becomes

$$(\mathcal{P}_\mu \beta^\mu + m)|\psi\rangle = 0 \tag{45}$$

Note that the $\mathcal{P}_\mu$ do not commute.

To derive the Hamiltonian for the particle in an electromagnetic field, we proceed in a manner similar to the free particle case. Multiply (45) on the left by $\mathcal{P}_\nu \beta^\nu \beta^\sigma$ to obtain

$$(\mathcal{P}_\nu \mathcal{P}_\mu \beta^\nu \beta^\sigma \beta^\mu + m\, \mathcal{P}_\nu \beta^\nu \beta^\sigma)|\psi\rangle = 0 \tag{46}$$

Eq. (46) can be written as

$$(\mathcal{P}_\mu \mathcal{P}_\nu \beta^\mu \beta^\sigma \beta^\nu + m\, \mathcal{P}_\nu \beta^\nu \beta^\sigma)|\psi\rangle = 0 \tag{47}$$

Add (46) and (47) to obtain

$$(\mathcal{P}_\nu \mathcal{P}_\mu \beta^\nu \beta^\sigma \beta^\mu + \mathcal{P}_\mu \mathcal{P}_\nu \beta^\mu \beta^\sigma \beta^\nu + 2m\, \mathcal{P}_\nu \beta^\nu \beta^\sigma)|\psi\rangle = 0$$

which can be written

$$\Big[\mathcal{P}_\nu \mathcal{P}_\mu (\beta^\nu \beta^\sigma \beta^\mu + \beta^\mu \beta^\sigma \beta^\nu) + 2m\, \mathcal{P}_\nu \beta^\nu \beta^\sigma + [\mathcal{P}_\mu, \mathcal{P}_\nu]\beta^\mu \beta^\sigma \beta^\nu\Big]|\psi\rangle = 0$$

$$\Big[\mathcal{P}_\nu \mathcal{P}_\mu (\beta^\nu g^{\sigma\mu} + \beta^\mu g^{\nu\sigma}) + 2m\, \mathcal{P}_\nu \beta^\nu \beta^\sigma + [\mathcal{P}_\mu, \mathcal{P}_\nu]\beta^\mu \beta^\sigma \beta^\nu\Big]|\psi\rangle = 0$$

$$\big(\mathcal{P}_\nu \mathcal{P}^\sigma \beta^\nu + \mathcal{P}^\sigma \mathcal{P}_\mu \beta^\mu + 2m\, \mathcal{P}_\nu \beta^\nu \beta^\sigma + [\mathcal{P}_\mu, \mathcal{P}_\nu]\beta^\mu \beta^\sigma \beta^\nu\big)|\psi\rangle = 0 \tag{48}$$

The first term in (48) can be written as

$$\mathcal{P}_\nu \mathcal{P}^\sigma \beta^\nu = [\mathcal{P}_\nu , \mathcal{P}^\sigma]\beta^\nu + \mathcal{P}^\sigma \mathcal{P}_\nu \beta^\nu$$

Thus (48) becomes

$$\begin{aligned}\Big(2\mathcal{P}^\sigma \mathcal{P}_\mu \beta^\mu + 2m\,\mathcal{P}_\nu \beta^\nu \beta^\sigma + [\mathcal{P}_\nu , \mathcal{P}^\sigma]\beta^\nu \\ + [\mathcal{P}_\mu , \mathcal{P}_\nu]\beta^\mu \beta^\sigma \beta^\nu\Big)|\psi\rangle = 0 \end{aligned} \tag{49}$$

In the first term of (49) we can put in $-m$ for $\mathcal{P}_\mu \beta^\mu$, so

(49) can be written

$$\begin{aligned}\mathcal{P}^\sigma|\psi\rangle &= \Big(\mathcal{P}_\nu \beta^\nu \beta^\sigma + \tfrac{1}{2m}[\mathcal{P}_\nu , \mathcal{P}^\sigma]\beta^\nu \\ &\qquad + \tfrac{1}{2m}[\mathcal{P}_\mu , \mathcal{P}_\nu]\beta^\mu \beta^\sigma \beta^\nu\Big)|\psi\rangle \\ &= \Big(\mathcal{P}_\nu \beta^\nu \beta^\sigma + \tfrac{1}{2m}[\mathcal{P}_\nu , \mathcal{P}_\mu]\beta^\nu g^{\mu\sigma} \\ &\qquad + \tfrac{1}{2m}[\mathcal{P}_\mu , \mathcal{P}_\nu]\beta^\mu \beta^\sigma \beta^\nu\Big)|\psi\rangle \\ &= \Big[\mathcal{P}_\nu \beta^\nu \beta^\sigma + \tfrac{1}{2m}[\mathcal{P}_\mu , \mathcal{P}_\nu]\big(\beta^\mu \beta^\sigma \beta^\nu + \beta^\mu g^{\nu\sigma}\big)\Big]|\psi\rangle\end{aligned}$$

For $\sigma = 0$,

$$\begin{aligned}\mathcal{P}^0|\psi\rangle &= \Big[\mathcal{P}_\nu \beta^\nu \beta^0 + \tfrac{1}{2m}[\mathcal{P}_\mu , \mathcal{P}_\nu]\big(\beta^\mu \beta^0 \beta^\nu + \beta^\mu g^{\nu 0}\big)\Big]|\psi\rangle \\ &= \Big[\mathcal{P}_0 \beta^{0^2} + \mathcal{P}_j \beta^j \beta^0 + \tfrac{1}{2m}[\mathcal{P}_\mu , \mathcal{P}_\nu]\big(\beta^\mu \beta^0 \beta^\nu \\ &\qquad + \beta^\mu g^{\nu 0}\big)\Big]|\psi\rangle\end{aligned} \tag{50}$$

Multiply (45) on the left by β^0 to obtain

$$\big(\mathcal{P}_0 \beta^{0^2} + \mathcal{P}_j \beta^0 \beta^j + m\beta^0\big)|\psi\rangle = 0$$

or

$$\mathcal{P}_0 \beta^{0^2}|\psi\rangle = \big(-\mathcal{P}_j \beta^0 \beta^j - m\beta^0\big)|\psi\rangle \tag{51}$$

Now put (51) into (50) to obtain

$$\begin{aligned}\mathcal{P}_0|\psi\rangle &= \Big[\mathcal{P}_j[\beta^j , \beta^0] - m\beta^0 + \tfrac{1}{2m}[\mathcal{P}_\mu , \mathcal{P}_\nu] \\ &\qquad \times \big(\beta^\mu \beta^0 \beta^\nu + \beta^\mu g^{0\nu}\big)\Big]|\psi\rangle\end{aligned} \tag{52}$$

or

$$\begin{aligned}P_0|\psi\rangle &= \Big[eA_0 + \mathcal{P}_j[\beta^j , \beta^0] - m\beta^0 + \tfrac{1}{2m}[\mathcal{P}_\mu , \mathcal{P}_\nu] \\ &\qquad \times \big(\beta^\mu \beta^0 \beta^\nu + \beta^\mu g^{0\nu}\big)\Big]|\psi\rangle\end{aligned} \tag{53}$$

Eq. (53) is analogous to (11) which was the free particle equation. Here, as for the free particle case, (53) is not equivalent to the Kemmer equation (45). To find the part of the Kemmer equation which was lost when (53) was derived, we can multiply (52) on the left by β^0 and subtract this from the Kemmer equation (45):

$$\mathcal{P}^0\beta^0|\psi\rangle = \Big[-\mathcal{P}_j\beta^{0^2}\beta^j - m\beta^{0^2} + \frac{1}{2m}\left[\mathcal{P}_0, \mathcal{P}_\nu\right]\beta^0\beta^\nu + \frac{1}{2m}\left[\mathcal{P}_\mu, \mathcal{P}_0\right]\beta^0\beta^\mu\Big]|\psi\rangle$$

$$= \left(-\mathcal{P}_j\beta^{0^2}\beta^j - m\beta^{0^2}\right)|\psi\rangle \tag{54}$$

Now subtract (54) from (45) to obtain

$$\left(-\mathcal{P}_j\beta^{0^2}\beta^j - m\beta^{0^2} + \mathcal{P}_j\beta^j + m\right)|\psi\rangle = 0 \tag{55}$$

Thus, (53) and (55) are together equivalent to (45).

Eq. (55) may be written

$$\left(\mathcal{P}_j\beta^j\beta^{0^2} - m\beta^{0^2} + m\right)|\psi\rangle = 0 \tag{56}$$

By using the fact that $\beta^0\beta^j\beta^0 = 0$, (56) can be written

$$\left(\mathcal{P}_j\left[\beta^j, \beta^0\right]\beta^0 - m\beta^{0^2} + m\right)|\psi\rangle = 0$$

or

$$\left[\left(\mathcal{P}_j\left[\beta^j, \beta^0\right] - m\beta^0\right)\beta^0 + m\right]|\psi\rangle = 0 \tag{57}$$

Thus, (57) is analogous to (18) for the free particle. So the two equations, (53) and (57), when taken together, are equivalent to the Kemmer equation for a particle in an electromagnetic field.

The right side of (53) contains the term $\left[\mathcal{P}_\mu, \mathcal{P}_\nu\right]$ which can be written

$$\left[\mathcal{P}_\mu, \mathcal{P}_\nu\right] = \left[P_\mu - eA_\mu, P_\nu - eA_\nu\right]$$

$$= [P_\mu, P_\nu] - e[P_\mu, A_\nu] - e[A_\mu, P_\nu] + e^2[A_\mu, A_\nu]$$

$$= -e[P_\mu, A_\nu] + e[P_\nu, A_\mu]$$

In the x-representation,

$$\langle x|[P_\mu, A_\nu]|\psi\rangle = \langle x|(P_\mu A_\nu - A_\nu P_\mu)|\psi\rangle$$

$$= i\partial_\mu \langle x|A_\nu|\psi\rangle - A_\nu(x)\langle x|P_\mu|\psi\rangle$$

$$= i\partial_\mu [A_\nu(x)\langle x|\psi\rangle] - i A_\nu(x)\partial_\mu \langle x|\psi\rangle$$

$$= \langle x|\psi\rangle\, i\partial_\mu A_\nu(x)$$

Thus

$$\langle x|[\mathcal{P}_\mu, \mathcal{P}_\nu]|\psi\rangle = -ie\langle x|\psi\rangle[\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)]$$

$$= -ie\langle x|\psi\rangle F_{\mu\nu}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor which has the following properties:

$$F_{\mu\nu} = -F_{\nu\mu}$$

$$F_{j0} = -E_j$$

$$F_{ij} = -B_k$$

where (i, j, k, cyclic)

$$(E_1, E_2, E_3) = (E_x, E_y, E_z)$$

and

$$(B_1, B_2, B_3) = (B_x, B_y, B_z)$$

Let us now define the antisymmetric tensor operator $G_{\mu\nu}$ by the equation,

$$[\mathcal{P}_\mu, \mathcal{P}_\nu] = -ie\, G_{\mu\nu}$$

so that

$$\langle x|G_{\mu\nu}|\psi\rangle = F_{\mu\nu}\langle x|\psi\rangle$$

Then (53) can now be written

$$P_0|\psi\rangle = \Big[eA_0 + \mathcal{P}_j[\beta^j, \beta^0] - m\beta^0 - \frac{ie}{2m} G_{\mu\nu}(\beta^\mu\beta^0\beta^\nu + \beta^\mu g^{0\nu})\Big]|\psi\rangle \tag{58}$$

The last term on the right side (58) can be symplified by using

$\beta^\mu\beta^\nu\beta^\sigma = 0$ for μ, ν, σ all different:

$$
\begin{aligned}
& G_{\mu\nu}\left(\beta^\mu\beta^0\beta^\nu + \beta^\mu g^{0\nu}\right) \\
&\quad = G_{0j}\left(\beta^{0^2}\beta^j + \beta^0 g^{0j}\right) + G_{j0}\left(\beta^j\beta^{0^2} + \beta^j g^{00}\right) \\
&\quad = G_{j0}\left(\beta^j\beta^{0^2} + \beta^j - \beta^{0^2}\beta^j\right) \\
&\quad = 2\, G_{j0}\, \beta^j\beta^{0^2}
\end{aligned}
$$

where we have used (2) in the last step.

Eq. (58) now becomes

$$
\begin{aligned}
P_0|\psi\rangle &= \left(eA_0 + P_j[\beta^j, \beta^0] - m\beta^0 - \frac{ie}{m} G_{j0}\beta^j\beta^{0^2}\right)|\psi\rangle \qquad (59) \\
&= \mathcal{H}'|\psi\rangle
\end{aligned}
$$

where the operator $\mathcal{H}'$ is the quantity in parentheses on the right side of (59). In matrix form

$$
\mathcal{H}' = \begin{bmatrix}
eA_0 & -P_1 & -P_2 & -P_3 & m \\
-P_1 & eA_0 & 0 & 0 & -\frac{ie}{m}G_{10} \\
-P_2 & 0 & eA_0 & 0 & -\frac{ie}{m}G_{20} \\
-P_3 & 0 & 0 & eA_0 & -\frac{ie}{m}G_{30} \\
m & 0 & 0 & 0 & eA_0
\end{bmatrix}
$$

We see that $\mathcal{H}'$ is not neo-Hermitian, so it cannot be taken as the Hamiltonian. Note also that $\mathcal{H}'$ is not Hermitian in the usual sense either, although in the free particle case H' was Hermitian. Since it is not neo-Hermitian for the same reason as H' was not neo-Hermitian in the free particle case, we can use the subsidiary condition (56) to construct a neo-Hermitian Hamiltonian in identically the same way as for the free particle case. Thus, the Hamiltonian for a particle in an electromagnetic field is given by

$$
\mathcal{H} = \mathcal{H}' + \frac{1}{m} P_j \beta^0\beta^j \left(P_i\beta^i\beta^{0^2} - m\beta^{0^2} + m\right).
$$

where we have used the subsidiary condition in the form (56).

When the expression for $\mathcal{H}'$ is inserted, $\mathcal{H}$ can be written

$$\mathcal{H} = eA_0 + \mathcal{P}_j \beta^j \beta^0 + \frac{1}{m} \mathcal{P}_i \mathcal{P}_j \beta^0 \beta^i \beta^j - m\beta^0 - \frac{ie}{m} G_{j0} \beta^j \beta^{0^2}$$

In matrix form,

$$\mathcal{H} = \begin{bmatrix} eA_0 & 0 & 0 & 0 & \frac{1}{m}(\underline{\mathcal{P}}^2 + m^2) \\ -\mathcal{P}_1 & eA_0 & 0 & 0 & -\frac{ie}{m} G_{10} \\ -\mathcal{P}_2 & 0 & eA_0 & 0 & -\frac{ie}{m} G_{20} \\ -\mathcal{P}_3 & 0 & 0 & eA_0 & -\frac{ie}{m} G_{30} \\ m & 0 & 0 & 0 & eA_0 \end{bmatrix}$$

where

$$\underline{\mathcal{P}}^2 = -\mathcal{P}_i \mathcal{P}^i$$

7. Spherically Symmetric Electromagnetic Field.

For a spherically symmetric electromagnetic field, $\underline{A} = 0$

so

$$\mathcal{H} = e\phi + P_j \beta^j \beta^0 + \frac{1}{m} P_i P_j \beta^0 \beta^i \beta^j$$
$$- m\beta^0 - \frac{ie}{m} G_{j0} \beta^j \beta^{0^2} \tag{60}$$

where we have set $\phi \equiv A_0$ which is the operator for the potential.

Also, for a spherically symmetric field,

$$i\, G_{j0} = [P_j, \phi]$$

In terms of the free particle Hamiltonian H,

$$\mathcal{H} = H + e\phi - \frac{ie}{m} G_{j0} \beta^j \beta^{0^2}$$

or in matrix form,

$$\mathcal{H} = \begin{bmatrix} e\phi & 0 & 0 & 0 & \frac{1}{m}(\underline{P}^2 + m^2) \\ -P_1 & e\phi & 0 & 0 & -\frac{ie}{m} G_{10} \\ -P_2 & 0 & e\phi & 0 & -\frac{ie}{m} G_{20} \\ -P_3 & 0 & 0 & e\phi & -\frac{ie}{m} G_{30} \\ m & 0 & 0 & 0 & e\phi \end{bmatrix}$$

The subsidiary condition (57) becomes identical to the free particle subsidiary condition when we specialize to a spherically symmetric potential. Thus the two equations,

$$P_0 |\psi\rangle = \mathcal{H} |\psi\rangle \tag{61}$$

and

$$(H' \beta^0 + m) |\psi\rangle = 0$$

where $\mathcal{H}$ is given by (60), are together equivalent to the Kemmer equation. Recall that the subsidiary condition can be written as

$$|\psi_i\rangle = -\frac{1}{m} P_i |\psi_4\rangle \qquad (i = 1, 2, 3) \tag{62}$$

For states $|\psi\rangle$ which are eigenstates of the Hamiltonian, we can write the eigenvalue equation,

$$\mathcal{H}|\psi\rangle = E|\psi\rangle \tag{63}$$

where E is the eigenvalue of $\mathcal{H}$.

In matrix form (63) may be written

$$\begin{bmatrix} E-e\phi & 0 & 0 & 0 & -\frac{1}{m}(\underline{P}^2+m^2) \\ P_1 & E-e\phi & 0 & 0 & \frac{ie}{m}G_{10} \\ P_2 & 0 & E-e\phi & 0 & \frac{ie}{m}G_{20} \\ P_3 & 0 & 0 & E-e\phi & \frac{ie}{m}G_{30} \\ -m & 0 & 0 & 0 & E-e\phi \end{bmatrix} \begin{bmatrix} |\psi_0\rangle \\ |\psi_1\rangle \\ |\psi_2\rangle \\ |\psi_3\rangle \\ |\psi_4\rangle \end{bmatrix} = 0 \tag{64}$$

The top and bottom rows of (64) yield, respectively,

$$(E-e\phi)|\psi_0\rangle - \frac{1}{m}(\underline{P}^2+m^2)|\psi_4\rangle = 0 \tag{65}$$

and

$$-m|\psi_0\rangle + (E-e\phi)|\psi_4\rangle = 0 \tag{66}$$

If we solve (66) for $|\psi_0\rangle$ and put the result into (65) we obtain

$$\left[(E-e\phi)^2 - \underline{P}^2 - m^2\right]|\psi_4\rangle = 0 \tag{67}$$

The solutions to (67) in the x-representation are given by Schiff for the case of a Coulomb field, and the energy levels are found. [2].

Except for a normalization factor, $|\psi_4\rangle$ is determined by (67). Eq. (66) gives $|\psi_0\rangle$ in terms of $|\psi_4\rangle$, and (62) gives $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$ in terms of $|\psi_4\rangle$.

For a Kemmer particle traveling in a spherically symmetric electromagnetic potential, the angular momentum is a constant of the motion. This can be shown as follows, where J_z is given by (35).

$$i\dot{J}_z = [J_z, \mathcal{H}] = \left[J_z, H + e\phi - \frac{ie}{m}G_{j0}\beta^j\beta^{0^2}\right]$$

We have previously shown that $\left[J_z , H\right] = 0$, so

$$
\begin{aligned}
i\dot{J}_z &= \left[J_z , e\phi - \frac{ie}{m} G_{j0}\beta^j\beta^{0^2}\right] \\
&= \left[J_z , e\phi\right] - \frac{ie}{m} G_{j0}\left[J_z , \beta^j\beta^{0^2}\right] \\
&\quad - \frac{ie}{m}\left[J_z , G_{j0}\right]\beta^j\beta^{0^2}
\end{aligned}
\tag{68}
$$

Now $\left[J_z , e\phi\right] = \left[L_z , e\phi\right] = 0$

because ϕ is spherically symmetric. Also,

$$
\begin{aligned}
G_{j0}\left[J_z , \beta^j\beta^{0^2}\right] &= G_{j0}\left[S_z , \beta^j\beta^{0^2}\right] \\
&= i\, G_{j0}\left[\left[\beta^1 , \beta^2\right] , \beta^j\beta^{0^2}\right] \\
&= i\, G_{j0}\left(\beta^1\beta^2\beta^j\beta^{0^2} - \beta^2\beta^1\beta^j\beta^{0^2}\right. \\
&\qquad \left. - \beta^j\beta^{0^2}\beta^1\beta^2 + \beta^j\beta^{0^2}\beta^2\beta^1\right) \\
&= i\, G_{j0}\left(\beta^1\beta^2\beta^j\beta^{0^2} - \beta^2\beta^1\beta^j\beta^{0^2}\right) \\
&= i\, G_{j0}\left[\left(-\beta^j\beta^2\beta^1 + \beta^1 g^{2j}\right)\beta^{0^2}\right. \\
&\qquad \left. - \left(-\beta^j\beta^1\beta^2 + \beta^2 g^{1j}\right)\beta^{0^2}\right] \\
&= i\, G_{j0}\left(g^{2j}\beta^1\beta^{0^2} - g^{1j}\beta^2\beta^{0^2}\right) \\
&= -i\, G_{20}\beta^1\beta^{0^2} + i\, G_{10}\beta^2\beta^{0^2} \\
&= -\left[P_2 , \phi\right]\beta^1\beta^{0^2} + \left[P_1 , \phi\right]\beta^2\beta^{0^2}
\end{aligned}
\tag{69}
$$

and

$$
\begin{aligned}
i\left[J_z , G_{j0}\right]\beta^j\beta^{0^2} &= i\left[L_z , G_{j0}\right]\beta^j\beta^{0^2} \\
&= \left[L_z , \left[P_j , \phi\right]\right]\beta^j\beta^{0^2} \\
&= \left[L_z , \left(P_j\phi - \phi P_j\right)\right]\beta^j\beta^{0^2} \\
&= \left[L_z , P_j\right]\phi\beta^j\beta^{0^2} - \phi\left[L_z , P_j\right]\beta^j\beta^{0^2}
\end{aligned}
\tag{70}
$$

Now, $$[L_z, P_j] = [X^1 P^2 - X^2 P^1, P_j]$$
$$= [X^1, P_j] P^2 - [X^2, P_j] P^1$$
$$= -i g^1{}_j P^2 + i g^2{}_j P^1$$
$$= i g^1{}_j P_2 - i g^2{}_j P_1$$

Thus, (70) becomes

$$i [J_z, G_{j0}] \beta^j \beta^{0^2} = i P_2 \phi \beta^1 \beta^{0^2} - i P_1 \phi \beta^2 \beta^{0^2} - i \phi P_2 \beta^1 \beta^{0^2} + i \phi P_1 \beta^2 \beta^{0^2}$$

$$= i [P_2, \phi] \beta^1 \beta^{0^2} - i [P_1, \phi] \beta^2 \beta^{0^2} \tag{71}$$

When we put (69) and (71) into (68) we see that $\dot{J}_z = 0$.
Similarly $\dot{J}_x = \dot{J}_y = 0$, so the angular momentum is a constant of the motion as required for a spherically symmetric potential.

8. Summary

We may summarize the arguments for taking H and $\mathcal{H}$ as the Hamiltonians as opposed to H' and $\mathcal{H}'$ (which Kemmer originally took to be the Hamiltonians) as follows:

The requirement that the expectation value of an observable be real leads to the condition that observables be what we have called "neo-Hermitian". Neither H' nor $\mathcal{H}'$ are neo-Hermitian, but H and $\mathcal{H}$ are. Furthermore, even if we had not introduced the requirement that an observable be neo-Hermitian, $\mathcal{H}'$ still presents a difficulty since it is not Hermitian even in the usual sense. Finally, H yields a velocity operator whose expectation value is consistent with the correspondence principle, while H' does not.

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